

Dirac equation in Kerr geometry

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Abstract. We are familiar with Dirac equation in flat space by which we can investigate the behaviour of half-integral spin particle. With the introduction of general relativistic effects the form of the Dirac equation will be modified. For the cases of different background geometry like Kerr, Schwarzschild etc. the corresponding form of the Dirac equation as well as the solution will be different. In 1972, Teukolsky wrote the Dirac equation in Kerr geometry. Chandrasekhar separated it into radial and angular parts in 1976. Later Chakrabarti solved the angular equation in 1984. In 1999 Mukhopadhyay and Chakrabarti have solved the radial Dirac equation in Kerr geometry in a spatially complete manner. In this review we will discuss these developments systematically and present some solutions.

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1. Introduction

Behaviour of particles with half integral spin can be investigated through the study of Dirac equation. Generally, we are familiar with the Dirac equation and its solution when the space-time is flat. In the curved space-time where the influence of the gravity is introduced, the corresponding equation will be changed in form. Its solution will also be different. In 1972, Teukolsky [1] wrote the Dirac equation in curved space-time particularly in Kerr geometry [2] using Newman-Penrose formalism [3]. Through this modified Dirac equation we can study the behaviour of spin half particles around the spinning black holes. Due to presence of central black hole the space-time is influenced and behaviour of the particle is changed with respect to that of flat space. From the same equation of Teukolsky, Dirac equation for Schwarzschild metric [2] (Schwarzschild geometry), where the central black hole is static can be studied

just by putting the angular momentum parameter a of the black hole to zero. So one can study how the behaviour of spin half particle in curved space time is influenced by the angular momentum of black hole. In 1976, Chandrasekhar [4] separated the Dirac equation in Kerr geometry into radial and angular parts and solved the radial part of the equation asymptotically. Chakrabarti in 1984 [5] solved the angular part analytically. Here we shall introduce the spatially complete analytical solution of radial Dirac equation [6-7]. So the the complete solution of Dirac equation can be studied. Far away from the black hole the the modified Dirac equation for curved space-time (for Kerr and Schwarzschild geometry [2-3]) and its solution reduce into that of the flat space.

In this review we will first indicate how Dirac equation in curved space-time can be written using Newman-Penrose formalism [3]. Newman-Penrose formalism is one of the tetrad formalism where null basis are chosen instead of orthonormal basis. To fulfill the understanding of Dirac equation in this formalism we also need to know the 'Spinor Analysis' [3]. In the next Section, we will briefly describe this in the context of our present purpose. In §3 we will write the Dirac equation in Newman-Penrose formalism for flat and curved space-time. For curved space we will separate the Dirac equation under the background of Kerr geometry. In §4 and §5 we will briefly outline the angular and radial solution of Dirac equation respectively. In §6 we make concluding remarks.

2. Spinor analysis

In Minkowski space we consider a point x^i ($i = 0, 1, 2, 3$) on a null ray whose norm is defined as

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0. \quad (1)$$

Now, we consider two complex numbers ξ^0 and ξ^1 , and their complex conjugate numbers $\bar{\xi}^{0'}$ and $\bar{\xi}^{1'}$ in terms of which each point can be written as,

$$x^0 = \frac{1}{\sqrt{2}}(\xi^0 \bar{\xi}^{0'} + \xi^1 \bar{\xi}^{1'}) \quad (2a)$$

$$x^1 = \frac{1}{\sqrt{2}}(\xi^0 \bar{\xi}^{1'} + \xi^1 \bar{\xi}^{0'}) \quad (2b)$$

$$x^2 = \frac{1}{\sqrt{2}}(\xi^0 \bar{\xi}^{1'} - \xi^1 \bar{\xi}^{0'}) \quad (2c)$$

$$x^3 = -\frac{i}{\sqrt{2}}(\xi^0 \bar{\xi}^{0'} - \xi^1 \bar{\xi}^{1'}) \quad (2d)$$

Conversely, we can write,

$$\xi^0 \bar{\xi}^{0'} = \frac{1}{\sqrt{2}}(x^0 + x^3) \quad (3a)$$

$$\xi^0 \bar{\xi}^{1'} = \frac{1}{\sqrt{2}}(x^1 + ix^2) \quad (3b)$$

$$\xi^1 \bar{\xi}^{0'} = \frac{1}{\sqrt{2}}(x^1 - ix^2) \quad (3c)$$

$$\xi^1 \bar{\xi}^{1'} = \frac{1}{\sqrt{2}}(x^0 - x^3) \quad (3d)$$

Let,

$$\xi_*^A = \alpha^A_B \xi^B, \quad (4a)$$

$$\bar{\xi}_*^{A'} = \bar{\alpha}_{B'}^{A'} \bar{\xi}^{B'} \quad (4b)$$

where, $(A, B, A', B' = 0, 1)$, are the linear transformations in complex two-dimensional spaces. The transformation of x^i is defined as,

$$x_*^i = \beta_j^i x^j. \quad (5)$$

Now, using equation (2) and (3) we can write,

$$\begin{aligned} x_*^0 &= \frac{1}{\sqrt{2}}(\alpha_0^0 \xi^0 + \alpha_1^0 \xi^1)(\bar{\alpha}_{0'}^{0'} \bar{\xi}^{0'} + \bar{\alpha}_{1'}^{0'} \bar{\xi}^{1'}) + \frac{1}{\sqrt{2}}(\alpha_0^1 \xi^0 + \alpha_1^1 \xi^1)(\bar{\alpha}_{0'}^{1'} \bar{\xi}^{0'} + \bar{\alpha}_{1'}^{1'} \bar{\xi}^{1'}) \\ &= \frac{1}{2}(\alpha_0^0 \bar{\alpha}_{0'}^{0'} + \alpha_0^1 \bar{\alpha}_{1'}^{0'})(x^0 + x^3) + \frac{1}{2}(\alpha_1^0 \bar{\alpha}_{0'}^{0'} + \alpha_1^1 \bar{\alpha}_{1'}^{0'})(x^0 - x^3) \\ &\quad + \frac{1}{2}(\alpha_0^0 \bar{\alpha}_{1'}^{0'} + \alpha_0^1 \bar{\alpha}_{1'}^{1'})(x^1 + ix^2) + \frac{1}{2}(\alpha_1^0 \bar{\alpha}_{0'}^{0'} + \alpha_1^1 \bar{\alpha}_{0'}^{1'})(x^1 - ix^2). \end{aligned} \quad (6)$$

Similarly, we can write down the relations between x_*^1 , x_*^2 and x_*^3 with α 's and x 's. Therefore, keeping in mind (5) we can write,

$$\beta_0^0 + \beta_3^0 = \alpha_0^0 \bar{\alpha}_{0'}^{0'} + \alpha_0^1 \bar{\alpha}_{1'}^{0'},$$

$$\beta_0^0 - \beta_3^0 = \alpha_1^0 \bar{\alpha}_{1'}^{0'} + \alpha_1^1 \bar{\alpha}_{1'}^{1'},$$

$$\beta_1^0 - i\beta_2^0 = \alpha_0^0 \bar{\alpha}_{1'}^{0'} + \alpha_0^1 \bar{\alpha}_{1'}^{1'},$$

$$\beta_1^0 + i\beta_2^0 = \alpha_1^0 \bar{\alpha}_{0'}^{0'} + \alpha_1^1 \bar{\alpha}_{0'}^{1'}.$$

Now, imposing the condition that the transformation (5) is Lorentzian we can write,

$$(\beta_0^0)^2 - (\beta_1^0)^2 - (\beta_2^0)^2 - (\beta_3^0)^2 = 1$$

So,

$$\left\| \begin{array}{cc} \alpha_0^0 \bar{\alpha}_0^{0'} + \alpha_0^1 \bar{\alpha}_0^{1'} & \alpha_0^0 \bar{\alpha}_1^{0'} + \alpha_0^1 \bar{\alpha}_1^{1'} \\ \alpha_1^0 \bar{\alpha}_0^{0'} + \alpha_1^1 \bar{\alpha}_0^{1'} & \alpha_1^0 \bar{\alpha}_1^{0'} + \alpha_1^1 \bar{\alpha}_1^{1'} \end{array} \right\| = 1. \quad (7)$$

This gives,

$$\left\| \begin{array}{cc} \alpha_0^0 & \alpha_1^0 \\ \alpha_0^1 & \alpha_1^1 \end{array} \right\| \left\| \begin{array}{cc} \bar{\alpha}_0^{0'} & \bar{\alpha}_1^{0'} \\ \bar{\alpha}_0^{1'} & \bar{\alpha}_1^{1'} \end{array} \right\| = \Delta \bar{\Delta} = 1 \quad (8)$$

Now we consider $\Delta = \bar{\Delta} = 1$, so individually each transformation of ξ is Lorentzian. So we can conclude if transformation (5) is Lorentzian, the necessary condition is transformation (4) is also Lorentzian.

Now we define spinors $\xi^A, \eta^{A'}$ of rank one as $\xi_*^A = \alpha_B^A \xi^B$ and $\eta_*^{A'} = \bar{\alpha}_{B'}^{A'} \eta^{B'}$, ($A, A', B, B' = 0$), where $\|\alpha_B^A\| = \|\bar{\alpha}_{B'}^{A'}\| = 1$. Since ξ^A and $\eta^{A'}$ are two spinors of same class,

$$\begin{array}{cc} \epsilon^0 & \epsilon^1 \\ \eta^0 & \eta^1 \end{array} = \xi^0 \eta^1 - \xi^1 \eta^0 \quad (9)$$

which is invariant under unimodular transformation, i.e.,

$$\epsilon_{AB} \xi^A \eta^B \rightarrow \text{invariant} \quad (10)$$

where, ϵ_{AB} is Levi-Civita symbol. Here as in the case of tensor analysis ϵ_{AB} and $\epsilon_{A'B'}$ are used to lower the spinor indices as, $\xi_A = \xi^C \epsilon_{CA}$.

Now, using above information the representation of position vector x^i can be written as

$$x^i \leftrightarrow \begin{array}{cc} \xi^0 \xi^{0'} & \xi^0 \bar{\xi}^{1'} \\ \xi^1 \bar{\xi}^{0'} & \xi^1 \bar{\xi}^{1'} \end{array} \quad \frac{1}{\sqrt{2}} \begin{array}{cc} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{array} \quad (11)$$

Generally any vector X^i can be written in terms of spinor of rank two as,

$$X^i \leftrightarrow \begin{array}{cc} \xi^{00'} & \xi^{01'} \\ \xi^{10'} & \xi^{11'} \end{array} \quad \frac{1}{\sqrt{2}} \begin{array}{cc} X^0 + X^3 & X^1 + iX^2 \\ X^1 - iX^2 & X^0 - X^3 \end{array} = X^{AB'} \quad (12)$$

So a 4-vector is associated with a hermitian matrix such that,

$$\begin{aligned} (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 &= (X^0 + X^3)(X^0 - X^3) - (X^1 + iX^2)(X^1 - iX^2) \\ &= 2(\xi^{00'} \bar{\xi}^{11'} - \xi^{01'} \bar{\xi}^{10'}) = (\xi^{00'} \xi_{00'} + \xi_{11'} \bar{\xi}^{11'} + \xi_{10'} \bar{\xi}^{10'} + \xi^{01'} \xi_{01'}) = X_{AB'} X^{AB'}. \end{aligned}$$

From the definition of norms, we can write it in two different representations:

$$g_{ij} X^i X^j = \epsilon_{AC} \epsilon_{B'D'} X^{AB'} X^{CD'} \quad (13)$$

Therefore, we can transform $X^i \leftrightarrow X^{AB'}$ using,

$$X^i = \sigma_{AB'}^i X^{AB'} \quad (14a)$$

$$X^{AB'} = \sigma_i^{AB'} X^i \quad (14b)$$

where, $\sigma_i^{AB'}$ and $\sigma_{AB'}^i$ are nothing but Pauli matrices and their conjugate matrices with a factor $\frac{1}{\sqrt{2}}$.

A curved space-time is locally Minkowskian. At each point of space-time an orthonormal Dyad basis can be set up as $\zeta_{(a)}^A$ and $\zeta_{(a')}^{A'}$ ($a, a' = 0, 1$ and $A, A' = 0, 1$) for spinors.

We define, $\zeta_{(0)}^A = o^A$ and $\zeta_{(1)}^A = l^A$. The condition of orthonormality is $\epsilon_{AB} o^A l^B = o^0 l^1 - o^1 l^0 = o_B l^B = -o^A l_A = 1$.

Also it is clear that, $\epsilon^{(a)(b)} \zeta_{(a)}^A \zeta_{(b)}^B = \epsilon^{AB}$.

Then the null vectors are defined as

$$l^i \leftrightarrow o^A \bar{o}^{B'}, m^i \leftrightarrow o^A \bar{l}^{B'}, \bar{m}^i \leftrightarrow l^A \bar{o}^{B'}, n^i \leftrightarrow l^A \bar{l}^{B'}.$$

Where, vectors obey relations of null tetrads such as,

$$l^i n_i = 1, m^i \bar{m}_i = -1 \text{ and all other products give zero.}$$

In this way using dyad basis we can set up four null vectors which are basis of Newman-Penrose formalism. Using (14) we can write the basis explicitly as

$$l^i = \sigma_{AB'}^i \zeta_{(0)}^A \bar{\zeta}_{(0')}^{B'} = \sigma_{AB'}^i o^A \bar{o}^{B'}, \quad (15a)$$

$$m^i = \sigma_{AB'}^i \zeta_{(0)}^A \bar{\zeta}_{(1')}^{B'} = \sigma_{AB'}^i o^A \bar{l}^{B'}, \quad (15b)$$

$$\bar{m}^i = \sigma_{AB'}^i \zeta_{(1)}^A \bar{\zeta}_{(0')}^{B'} = \sigma_{AB'}^i l^A \bar{o}^{B'}, \quad (15c)$$

$$n^i = \sigma_{AB'}^i \zeta_{(1)}^A \bar{\zeta}_{(1')}^{B'} = \sigma_{AB'}^i l^A \bar{l}^{B'}. \quad (15d)$$

Thus, in Newman-Penrose formalism the Pauli matrices change their forms as,

$$\sigma_{AB'}^i = \frac{1}{\sqrt{2}} \begin{vmatrix} l^i & m^i \\ \bar{m}^i & n^i \end{vmatrix} \quad (16a)$$

$$\sigma_i^{AB'} = \frac{1}{\sqrt{2}} \begin{vmatrix} n_i & -\bar{m}_i \\ -\bar{m}_i & l_i \end{vmatrix}. \quad (16b)$$

Therefore in this basis, the directional derivatives can be written as,

$$D = l^i \partial_i, \underline{\Delta} = n^i \partial_i, \delta = m^i \partial_i \text{ and } \delta^* = \bar{m}^i \partial_i.$$

Thus, the spinor equivalents of Newman-Penrose formalism are

$$\partial_{00'} = D, \partial_{11'} = \underline{\Delta}, \partial_{01'} = \delta, \partial_{10'} = \delta^*.$$

Due to the reason, as explained earlier $\nabla_i \leftrightarrow \nabla_{AB'}$, so we can write,

$$\nabla_i X_j = X_{j;i} \leftrightarrow \nabla_{AB'} X_{CD'} = X_{CD';AB'},$$

therefore,

$$X_{CD';AB'} = \sigma_{CD'}^j \sigma_{AB'}^i X_{j;i}. \quad (17)$$

For covariant derivatives spin coefficients Γ are introduced. In the Newman-Penrose formalism these different coefficients are assigned in terms of special symbols which are given below:

$$\Gamma_{(a)(b)(c)(d')}$$

(a)(b)	00	01 or 10	11
(c)(d')			
00'	κ	ε	π
10'	ρ	α	λ
01'	σ	β	μ
11'	τ	γ	ν

3. Dirac equation and its separation

Before going into discussion, we should mention about the unit of the system. Here we have chosen throughout $\hbar = c = G = 1$, where \hbar = Planck constant, c = speed of light and G = gravitational constant. It is very clear that simultaneously all these quantities are chosen as unity implying the corresponding system is dimensionless.

The Dirac equation in flat space using Newman-Penrose formalism can be written as,

$$\sigma_{AB'}^i \partial_i P^A + i\mu_* \bar{Q}_{B'} = 0 \quad (18a)$$

$$\sigma_{AB'}^i \partial_i Q^A + i\mu_* \bar{P}_{B'} = 0. \quad (18b)$$

Here, P^A and $\bar{Q}^{A'}$ are the pair of spinors. $\mu_*/\sqrt{2}$ is the mass of the particles and $\sigma_{AB'}^i$ is nothing but Pauli matrix, because $1/\sqrt{2}$ factors are canceled in the equation.

In curved space time Dirac equation reduces to

$$\sigma_{AB'}^i P_{;i}^A + i\mu_* \bar{Q}^{C'} \epsilon_{C'B'} = 0, \quad (19a)$$

$$\sigma_{AB'}^i Q_{;i}^A + i\mu_* \bar{P}^{C'} \epsilon_{C'B'} = 0, \quad (19b)$$

where, $\sigma_{AB'}^i$ is same as defined in equation (16a).

Now, consider $B' = 0$, then (19a) reduces to

$$\sigma_{00'}^i P_{;i}^0 + \sigma_{10'}^i P_{;i}^1 - i\mu_* \bar{Q}^{1'} = 0$$

or,

$$(\partial_{00'} P^0 + \Gamma_{00'}^0 P^b) + (\partial_{10'} P^1 + \Gamma_{010'}^0 P^b) - i\mu_* \bar{Q}^{1'} = 0,$$

Therefore,

$$(D + \Gamma_{1000'} - \Gamma_{0010'}) P^0 + (\delta^* + \Gamma_{1100'} - \Gamma_{0110'}) P^1 - i\mu_* \bar{Q}^{1'} = 0 \quad (20)$$

Similarly, choosing $B' = 1$, we can get another similar type equation and then we can get corresponding conjugate equation of both by interchanging P and Q . Now choosing,

$$F_1 = P^0, F_2 = P^1, G_1 = \bar{Q}^{1'}, G_2 = -\bar{Q}^{0'}$$

and replacing various spin coefficients by their named symbols we get the the Dirac equation in Newman-Penrose formalism in its reduced form as,

$$(D + \varepsilon - \rho)F_1 + (\delta^* + \pi - \alpha)F_2 = i\mu_*G_1, \quad (21a)$$

$$(\underline{\Delta} + \mu - \gamma)F_2 + (\delta + \beta - \tau)F_1 = i\mu_*G_2, \quad (21b)$$

$$(D + \varepsilon^* - \rho^*)G_2 - (\delta + \pi^* - \alpha^*)G_1 = i\mu_*F_2, \quad (21c)$$

$$(\underline{\Delta} + \mu^* - \gamma^*)G_1 - (\delta^* + \beta^* - \tau^*)G_2 = i\mu_*F_1. \quad (21d)$$

3.1. Basis vectors of Newman-Penrose formalism in terms of Kerr geometry

The contravariant form of Kerr metric is given as [3],

$$g^{ij} = \begin{pmatrix} \Sigma^2/\rho^2\Delta & 0 & 0 & 2aMr/\rho^2\Delta \\ 0 & -\Delta/\rho^2 & 0 & 0 \\ 0 & 0 & -1/\rho^2 & 0 \\ 2aMr/\rho^2\Delta & 0 & 0 & -(\Delta - a^2\sin^2\theta)/\rho^2\Delta\sin^2\theta \end{pmatrix} \quad (22)$$

where, E is the energy, a is specific angular momentum of the black hole, M = mass of the black hole, $\rho^2 = r^2 + a^2\cos^2\theta$ (should not confuse with the spin coefficient $\Gamma_{(0)(0)(1)(0)} = \rho$), $\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$, $\Delta = r^2 + a^2 - 2Mr$.

In Kerr geometry, the tangent vectors of null geodesics are: $\frac{dt}{d\tau} = \frac{(r^2+a^2)}{\Delta}E$, $\frac{dr}{d\tau} = \pm E$, $\frac{d\theta}{d\tau} = 0$, $\frac{d\phi}{d\tau} = \frac{a}{\Delta}E$, where τ is the proper time (not to be confused with spin coefficient $\Gamma_{(0)(0)(1)(1)} = \tau$).

Now, the basis of Newman-Penrose formalism can be defined in Kerr geometry as (in tetrad form),

$$l_i = \frac{1}{\Delta}(\Delta, -\rho^2, 0, -a\Delta\sin^2\theta), \quad (23a)$$

$$n_i = \frac{1}{2\rho^2}(\Delta, \rho^2, 0, -a\Delta\sin^2\theta), \quad (23b)$$

$$m_i = \frac{1}{\bar{\rho}\sqrt{2}}(ia\sin\theta, 0, -\rho^2, -i(r^2 + a^2)\sin\theta), \quad (23c)$$

$$l^i = \frac{1}{\Delta}(r^2 + a^2, \Delta, 0, a), \quad (23d)$$

$$n^i = \frac{1}{\bar{\rho}\sqrt{2}}(r^2 + a^2, -\Delta, 0, a), \quad (23e)$$

$$m_i = \frac{1}{\bar{\rho}\sqrt{2}}(iasin\theta, 0, 1, icosec\theta), \quad (23f)$$

\bar{m}_i and \bar{m}^i are nothing but complex conjugates of m_i and m^i respectively.

3.2. Separation of Dirac equation into radial and angular parts

It is clear that the basis vectors basically become derivative operators when these are applied as tangent vectors to the function $e^{i(\sigma t + m\phi)}$. Here, σ is the frequency of the particle (not to be confused with spin coefficient $\Gamma_{(0)(0)(0)(1')} = \sigma$) and m is the azimuthal quantum number [3].

Therefore, we can write,

$$\vec{l} = D = \mathcal{D}_0, \quad \vec{n} = \underline{\Delta} = -\frac{\Delta}{2\rho^2}\mathcal{D}_0^\dagger, \quad \vec{m} = \delta = \frac{1}{\bar{\rho}\sqrt{2}}\mathcal{L}_0^\dagger, \quad \vec{\bar{m}} = \delta^* = \frac{1}{\bar{\rho}^*\sqrt{2}}\mathcal{L}_0.$$

where,

$$\mathcal{D}_n = \partial_r + \frac{iK}{\Delta} + 2n\frac{r-M}{\Delta}, \quad (24a)$$

$$\mathcal{D}_n^\dagger = \partial_r - \frac{iK}{\Delta} + 2n\frac{r-M}{\Delta}, \quad (24b)$$

$$\mathcal{L}_n = \partial_\theta + Q + n\cot\theta \quad (25a)$$

$$\mathcal{L}_n^\dagger = \partial_\theta - Q + n\cot\theta. \quad (25b)$$

$$K = (r^2 + a^2)\sigma + am, \quad Q = a\sigma\sin\theta + m\csc\theta.$$

The spin coefficients can be written as combination of basis vectors in Newman-Penrose formalism which are now expressed in terms of elements of different components of Kerr metric. So we are combining those different components of basis vectors in a suitable manner and get the spin coefficients as,

$$\kappa = \sigma = \lambda = \nu = \varepsilon = 0. \quad (26a)$$

$$\bar{\rho} = -\frac{1}{\bar{\rho}^*}, \quad \beta = \frac{\cot\theta}{\bar{\rho}^*2\sqrt{2}}, \quad \pi = \frac{iasin\theta}{(\bar{\rho}^*)^2\sqrt{2}},$$

$$\tau = -\frac{iasin\theta}{\rho^2\sqrt{2}}, \quad \mu = -\frac{\Delta}{2\rho^2\bar{\rho}^*}, \quad \gamma = \mu + \frac{r-M}{2\rho^2}, \quad \alpha = \pi - \beta^*. \quad (26b)$$

Using the above definitions and results and choosing $f_1 = \bar{\rho}^*F_1$, $g_2 = \bar{\rho}G_2$, $f_2 = F_2$, $g_1 = G_1$ the Dirac equation is reduced to

$$\mathcal{D}_0f_1 + 2^{-1/2}\mathcal{L}_{1/2}f_2 = (i\mu_*r + a\mu_*\cos\theta)g_1, \quad (27a)$$

$$\Delta\mathcal{D}_{1/2}^\dagger f_2 - 2^{1/2}\mathcal{L}_{1/2}^\dagger f_1 = -2(i\mu_*r + a\mu_*\cos\theta)g_2, \quad (27b)$$

$$\mathcal{D}_0g_2 - 2^{-1/2}\mathcal{L}_{1/2}^\dagger g_1 = (i\mu_*r - a\mu_*\cos\theta)f_2, \quad (27c)$$

$$\Delta\mathcal{D}_{1/2}^\dagger g_1 + 2^{1/2}\mathcal{L}_{1/2}g_2 = -2(i\mu_*r - a\mu_*\cos\theta)f_1, \quad (27d)$$

Now we will separate the Dirac equation into radial and angular parts by choosing,

$$f_1(r, \theta) = R_{-1/2}(r)S_{-1/2}(\theta), \quad f_2(r, \theta) = R_{1/2}(r)S_{1/2}(\theta), \\ g_1(r, \theta) = R_{1/2}(r)S_{-1/2}(\theta), \quad g_2(r, \theta) = R_{-1/2}(r)S_{1/2}(\theta).$$

Replacing these f_i and g_i ($i = 1, 2$) into (27) and using separation constant λ we get,

$$\mathcal{L}_{\frac{1}{2}} S_{+\frac{1}{2}} = -(\lambda - am_p \cos \theta) S_{-\frac{1}{2}} \quad (28a)$$

$$\mathcal{L}_{\frac{1}{2}}^\dagger S_{-\frac{1}{2}} = +(\lambda + am_p \cos \theta) S_{+\frac{1}{2}} \quad (28b)$$

$$\Delta^{\frac{1}{2}} \mathcal{D}_0 R_{-\frac{1}{2}} = (\lambda + im_p r) \Delta^{\frac{1}{2}} R_{+\frac{1}{2}}, \quad (29a)$$

$$\Delta^{\frac{1}{2}} \mathcal{D}_0^\dagger \Delta^{\frac{1}{2}} R_{+\frac{1}{2}} = (\lambda - im_p r) R_{-\frac{1}{2}}, \quad (29b)$$

where, m_p is the mass of the particle which is nothing but $2^{1/2}\mu_*$. Also, $2^{1/2}R_{-1/2}$ is redefined as $R_{-1/2}$.

Equations (28) and (29) are the angular and radial Dirac equation respectively in coupled form with the separation constant λ [3].

4. Solution of angular Dirac equation

Decoupling equation (28) we obtain the eigenvalue equation for spin- $\frac{1}{2}$ particles as

$$\left[\mathcal{L}_{\frac{1}{2}} \mathcal{L}_{\frac{1}{2}}^\dagger + \frac{am_p \sin \theta}{\lambda + am_p \cos \theta} \mathcal{L}_{\frac{1}{2}}^\dagger + (\lambda^2 - a^2 m_p^2 \cos^2 \theta) \right] S_{-\frac{1}{2}} = 0. \quad (30)$$

Similarly, one can obtain decoupled equation for spin- $+\frac{1}{2}$ particles. Here, the separation constant λ is considered to be the eigenvalue of the equation. The exact solutions of this equation for λ and $S_{-\frac{1}{2}}$ is possible in terms of orbital angular momentum quantum number l and the spin of the particle s when the parameter $\rho_1 = \frac{m_p}{\sigma} = 1$. When the angular momentum of the black hole is zero i.e., Schwarzschild case, the equation is reduced in such a form that whose solution is nothing but standard spherical harmonics such as [8-9],

$$S_{-1/2}(\theta)e^{im\phi} = {}_{-\frac{1}{2}}Y_{lm}(\theta, \phi), \quad (31)$$

the eigenvalue i.e., the separation constant can be solved as,

$$\lambda^2 = (l + 1/2)^2. \quad (32)$$

Similarly, for spin- $+\frac{1}{2}$ particle one can solve $S_{+1/2}$ as

$$S_{+1/2}(\theta)e^{im\phi} = {}_{+\frac{1}{2}}Y_{lm}(\theta, \phi), \quad (33)$$

with same eigenvalue λ .

For any non-integral, massless, spin particle the solutions are [8-9]

$$S_{\pm s}(\theta)e^{im\phi} = {}_{\pm s}Y_{lm}(\theta, \phi), \quad (34)$$

$$\lambda^2 = (l + |s|)(l - |s| + 1). \quad (35)$$

In the case of Kerr geometry, when $a \neq 0$ the equation can be solved by perturbative procedure [5] with perturbative parameter $a\sigma$. The solution for $\rho_1 = m_p/\sigma = 1$ and $s = \pm \frac{1}{2}$ is [5]

$$\lambda^2 = (l + \frac{1}{2})^2 + a\sigma(p + 2m) + a^2\sigma^2 \left[1 - \frac{y^2}{2(l+1) + a\sigma x} \right], \quad (36)$$

$$\frac{1}{2}S_{lm} = \frac{1}{2}Y_{lm} + \frac{a\sigma y}{2(l+1) + a\sigma x} \frac{1}{2}Y_{l+1m} \quad (37a)$$

$$-\frac{1}{2}S_{lm} = -\frac{1}{2}Y_{lm} - \frac{a\sigma y}{2(l+1) + a\sigma x} -\frac{1}{2}Y_{l+1m} \quad (37b)$$

where,

$$p = F(l, l); \quad x = F(l+1, l+1); \quad y = F(l, l+1) \quad (38)$$

and

$$F(l_1, l_2) = [(2l_2 + 1)(2l_1 + 1)]^{\frac{1}{2}} \langle l_2 1 m 0 | l_1 m \rangle [\langle l_2 1 \frac{1}{2} 0 | l_1 \frac{1}{2} \rangle + (-1)^{l_2-l} \langle l_2 1 m 0 | l_1 m \rangle [\langle l_2 1 \frac{1}{2} 0 | l_1 \frac{1}{2} \rangle + (-1)^{l_2-l} \rho_1 \sqrt{2} \langle l_2 1 -\frac{1}{2} 1 | l_1 \frac{1}{2} \rangle]]. \quad (39)$$

with $\langle \dots | \dots \rangle$ are the usual Clebsh-Gordon coefficients.

If $\rho_1 \neq 1$ then exact solution is not possible. In those cases the analytic expression of eigenvalue and angular wave-function are found as infinite series not in a compact form as the case $\rho_1 = 1$.

From the general convergence of series expansions one can truncate the infinite series upto certain order for particular values of l , s and m . For $l = \frac{1}{2}$, $s = -\frac{1}{2}$ and $m = -\frac{1}{2}$, up to third order in $a\sigma$, one obtains [5],

$$\begin{aligned} \lambda^2 &= (l + \frac{1}{2})^2 + a\sigma f_1(l, m) + (a\sigma)^2 f_2(l, m) + (a\sigma)^3 f_3(l, m), \quad (40) \\ -\frac{1}{2}S_{\frac{1}{2}-\frac{1}{2}} &= -\sin\theta - \left(\sin^3 \frac{\theta}{2} - \sin\theta \cos \frac{\theta}{2} \right) \left[\frac{2}{3}a\sigma(1 + \rho_1) + \frac{4}{15}(a\sigma)^2(1 - \rho_1^2) \right. \\ &\quad \left. + \frac{2}{5}(a\sigma)^2(1 - \rho_1^2) \left[\sin^5 \frac{\theta}{2} - 6\sin^2 \frac{\theta}{2} \cos^3 \frac{\theta}{2} + 3\sin \frac{\theta}{2} \cos^4 \frac{\theta}{2} \right] \right]. \quad (41) \end{aligned}$$

The accuracy of eigenvalues and eigenfunctions decreases as $a\sigma \rightarrow 1$.

5. Solution of radial Dirac equation

In the radial equation independent variable r is extended from 0 to ∞ . For mathematical simplicity we change the independent variable r to r_* as

$$r_* = r + \frac{2Mr_+ + am/\sigma}{r_+ - r_-} \log \left(\frac{r}{r_+} - 1 \right) - \frac{2Mr_- + am/\sigma}{r_+ - r_-} \log \left(\frac{r}{r_-} - 1 \right) \quad (42)$$

(for $r > r_+$), here in new r_* co-ordinate system horizon r_+ is shifted to $-\infty$ unless $\sigma \leq -\frac{am}{2Mr_+}$ [3], so the region is extended from $-\infty$ to ∞ . We also choose $R_{-\frac{1}{2}} = P_{-\frac{1}{2}}$, $\Delta^{\frac{1}{2}}R_{+\frac{1}{2}} = P_{+\frac{1}{2}}$. Then we are defining

$$(\lambda \pm im_p r) = \exp(\pm i\theta) \sqrt{(\lambda^2 + m_p^2 r^2)}$$

and

$$P_{+\frac{1}{2}} = \psi_{+\frac{1}{2}} \exp \left[-\frac{1}{2} i \tan^{-1} \left(\frac{m_p r}{\lambda} \right) \right],$$

$$P_{-\frac{1}{2}} = \psi_{-\frac{1}{2}} \exp \left[+\frac{1}{2} i \tan^{-1} \left(\frac{m_p r}{\lambda} \right) \right].$$

Finally choosing,

$$Z_{\pm} = \psi_{+\frac{1}{2}} \pm \psi_{-\frac{1}{2}}$$

and combining the differential equations (29) we get,

$$\left(\frac{d}{d\hat{r}_*} - W \right) Z_+ = i\sigma Z_-, \quad (43a)$$

and

$$\left(\frac{d}{d\hat{r}_*} + W \right) Z_- = i\sigma Z_+, \quad (43b)$$

where,

$$\hat{r}_* = r_* + \frac{1}{2\sigma} \tan^{-1} \left(\frac{m_p r}{\lambda} \right)$$

and

$$W = \frac{\Delta^{\frac{1}{2}}(\lambda^2 + m_p^2 r^2)^{3/2}}{\omega^2(\lambda^2 + m_p^2 r^2) + \lambda m_p \Delta / 2\sigma}. \quad (44)$$

where, $\omega^2 = \frac{K}{\sigma}$.

Now decoupling equations (43a-b) we get,

$$\left(\frac{d^2}{d\hat{r}_*^2} + \sigma^2 \right) Z_{\pm} = V_{\pm} Z_{\pm}. \quad (45)$$

where, $V_{\pm} = W^2 \pm \frac{dW}{d\hat{r}_*}$ and \hat{r}_* is extended from $-\infty$ (horizon) to $+\infty$.

The equation (45) is nothing but one dimensional Schrödinger equation [10] with potentials V_{\pm} and the energy of the particle σ^2 (since the system is dimensionless) in Cartesian co-ordinate system. The equation (45) can be solved by WKB approximation method [10-11]. The corresponding solution is [6-7],

$$Z_{\pm} = \frac{A_{\pm}}{\sqrt{k_{\pm}}} \exp(iu_{\pm}) \pm \frac{B_{\pm}}{\sqrt{k_{\pm}}} \exp(-iu_{\pm}) \quad (46)$$

where,

$$k_{\pm} = \sqrt{(\sigma^2 - V_{\pm})}, \quad (47)$$

and

$$u_{\pm} = \int k_{\pm} d\hat{r}_*. \quad (48)$$

Now we improve the solution by introducing space dependences on coefficients A_{\pm} and B_{\pm} [6-7] (this is beyond WKB approximation, because WKB deals with solutions with constant coefficients). It is seen that far away from a black hole, potential varies very slowly. Thus, in those regions one can safely write,

$$A_{\pm} - B_{\pm} = \text{Constant}(= c). \quad (49)$$

Since the sum of reflection and transmission coefficients must be unity,

$$A_{\pm}^2 + B_{\pm}^2 = k_{\pm}. \quad (50)$$

Near the horizon it is seen that potential height reduces to zero so the reflection in that region is almost zero and transmission is almost 100%. This is the *inner boundary condition*. Solving (49) and (50) we get analytical expression of space dependent reflection and transmission coefficients far away from the black hole which satisfy *outer boundary condition*. Combining the inner and outer boundary conditions, we get analytical expression of space dependent coefficient A_{\pm} and B_{\pm} which is valid in whole region $(-\infty \text{ to } +\infty)$. For details see [6-7]. The space dependency of A_{\pm} and B_{\pm} i.e. the transmission and reflection coefficients arises due to the variation of potential with distance. So from the analytical expressions one can easily find out at each point what fraction of incoming matter is going inward and what other fraction is going outward as a result of the interaction with the black hole. These space dependent transmission and reflection coefficients are given below [6-7],

$$T_{\pm} = \frac{a_{\pm}^2}{k_{\pm}} = \frac{(c_1 + \frac{c}{2})}{h_{\pm}} \left(c_1 + \frac{c}{2} + \sqrt{2k_{\pm} - c^2} \right) + \frac{2k_{\pm} - c^2}{4h_{\pm}} \quad (51a)$$

$$R_{\pm} = \frac{b_{\pm}^2}{k_{\pm}} = \frac{(c_2 - \frac{c}{2})}{h_{\pm}} \left(c_2 - \frac{c}{2} + \sqrt{2k_{\pm} - c^2} \right) + \frac{2k_{\pm} - c^2}{4h_{\pm}}. \quad (51b)$$

Here, a_{\pm} and b_{\pm} are defined as

$$a_{\pm} = \frac{A_{\pm}}{\sqrt{h_{\pm}/k_{\pm}}}, \quad (52a)$$

$$b_{\pm} = \frac{B_{\pm}}{\sqrt{h_{\pm}/k_{\pm}}} \quad (52b)$$

which are transmitted and reflected amplitudes of the solution with modified WKB method (going beyond WKB method) and

$$h_{\pm} = \left(c_1 + \frac{c}{2}\right)^2 + \left(c_2 - \frac{c}{2}\right)^2 + (c_1 + c_2)\sqrt{2k_{\pm} - c^2} + \frac{(2k_{\pm} - c^2)}{2}, \quad (53)$$

where, c_1 and c_2 are two constants introduced to satisfy the inner boundary condition. The final form of the solution is

$$Z_{\pm} = \frac{a_{\pm}}{\sqrt{k_{\pm}}} \exp(iu_{\pm}) \pm \frac{b_{\pm}}{\sqrt{k_{\pm}}} \exp(-iu_{\pm}). \quad (54)$$

Since the relation between Z_{\pm} and $R_{\pm\frac{1}{2}}$ is known, one can easily calculate the radial wave function $R_{\pm\frac{1}{2}}$.

6. Conclusions

In this review we write the Dirac equation in curved space-time and particularly in Kerr geometry. From this, the behaviour of non-integral spin particles can be studied in curved space-time. From the form of the equation and its solution it is clear that in curved space the particles behave differently than in a flat space-time. The Newman-Penrose formalism is used to write the equation where the basis system is null. Dirac equation is separated into angular and radial parts. Similar separation can be possible on the background of Dyon black hole [12]. The solution of angular component of the Dirac equation is first reviewed. The exact solution is possible for $\frac{m_p}{\sigma} = 1$, otherwise the solution is approximate [5]. Unlike in the case of a Kerr black hole, the solution of the angular equation around a Schwarzschild black hole is independent of the azimuthal or meridional angles [5, 13, 14]. This is expected because of symmetry of the space-time.

The radial Dirac equation is solved using WKB approximation more clearly modified WKB approximation [6-7], where the space dependent transmission and reflection coefficients are calculated. Although WKB method is an approximate method, it is improvised in such a way that spatial dependence of the coefficients of the wave function is obtained. This way we ensure that the analytical solution is closer to the exact solution. The reflection and transmission coefficients were found to distinguish strongly the solutions of different rest masses and different energies. The solution might be of immense use in the study of the spectrum of particles emitted from a black hole horizon (Hawking radiation).

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